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Probability Approximation for Compound Binomial and Compound Poisson Collective Risk Models

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Abstract

The collective risk model holds significant importance in decision-making and risk management contexts. It primarily focuses on the aggregation of random sums of random variables. In this work, we investigate the approximation of both compound binomial and compound Poisson collective risk models, where the number of claims follows binomial and Poisson distributions, respectively. Using the characteristic function as the primary analytical tool, we derive refined error bounds. Our results are established under the assumption of finite $(2+\delta)$-th moment, where $0 < \delta \leq 1$. Furthermore, we improve existing error bounds, particularly in regimes where the model parameters are sufficiently large.

Keywords: compound binomial distribution; compound Poisson distribution; random sums; collective risk model; characteristic function.

1 Introduction

In the realm of actuarial science and risk modeling, the calculation of aggregate losses and their corresponding probabilities plays a pivotal role in facilitating informed decision-making and proficient risk management. The known distributions of both the frequency and severity of claims enable the computation of total loss, albeit this procedure may be viewed as routine [5]. Consequently, the utilization of probability approximation becomes a valuable strategy for evaluating the chance and amount of claim payouts for insurance companies.

The aggregate losses in actuarial science can be characterized through two distinct models: the individual risk model and the collective risk model. In the individual risk model, aggregate losses depict the cumulative sum of losses from each individual contract, delimited by restrictions on the frequency of claims. An illustrative example of this model is a life insurance policy with a singular claim. Conversely, the collective risk model entails the aggregation of losses from only one insured. The frequency of claims, reflective of the quantity of losses, exhibits a stochastic nature over time, as evidenced in insurance domains like health and automobile coverage. In addition, the amount of each claim may vary significantly, differing from the structure observed in life insurance.

This study focuses on the collective risk model, emphasizing its complexity in contrast to the individual risk model. This distinction arises from the random nature of the number of claims in the collective risk model, in contrast to the fixed number of claims in another model. Notably, in light of the collective risk model's contractual nature, which is typically limited to a short policy period, often a year with the possibility of renewal, the model does not incorporate the discount factor of interest [12].

To introduce an aggregate loss for the collective risk model, let $X_1, X_2, X_3, \ldots$ be independent and identically distributed positive integer–valued random variables with the same distribution as X, denoted by $X \stackrel{\mathrm{d}}{=} X_1$, where $\stackrel{\mathrm{d}}{=}$ signifies equal in distribution. It is noteworthy that the identically distributed condition holds, as we are considering the claim behavior for individuals. Additionally, we assume that the random variables are positive, given that $X = 0$ signifies the absence of a claim. An aggregate loss S_N for the collective risk model is a sum of N random variables defined by,

$$S_N = \sum_{i=1}^{N} X_i,$$

where N is a positive integer-valued random variable representing the number of claims and X_i is the severity of the ith claim. For convention, we let $S_N = 0$ for $N = 0$.

As mentioned above, the distribution of the total loss can be computed when the distributions of frequency and severity of claims are known. Alternatively, probability approximations provide a practical way to estimate these distributions. To evaluate the accuracy of such approximations, one may refer to the Berry–Esseen bound, developed by Berry [2] and Esseen [3], for non-random sums of independent random variables. For random sums, the Berry–Esseen bounds were proposed by Korolev and Dorofeeva [9] using the characteristic function method. Further developments by Sunklodas [16, 17] combined this approach with Stein's method [15] to derive bounds for normal approximation. Notably, Stein's method also applies to random variables that are not sums, such as in the discretized normal approximation [14]. In some cases, the normal distribution may not be suitable [10].

Another form of approximation error bound arises from the local limit theorem, which provides an estimate for approximating point probabilities using a known probability density function. The development of this theorem is reviewed in [11], dependent sums [1], independent lattice-valued sums [6], and independent and identically distributed sums [13]. A version of the local limit theorem for random sums was established by Kongjiw et al. [8] in 2023.

In this study, our attention is directed towards situations in which the number of claims, denoted as N, adheres to both binomial and Poisson distributions. Specifically, we refer to a random sum S_N based on the distribution of N. For instance, S_N displays a compound binomial distribution and a compound Poisson distribution when N follows binomial and Poisson distributions, respectively. The bounds in the local limit theorem for compound binomial and compound Poisson collective risk models $P(S_N = k)$ for $k \in \mathbb{N}$ are enhanced. The primary tool used for deriving these approximations is the characteristic function method.

The remainder of this paper is organized as follows: Section 2 presents the bounds for the compound binomial collective risk model, while Section 3 examines the compound Poisson collective risk model. Additionally, the results for both models are investigated under the condition of finite $(2+\delta)$-th moment, where $0 < \delta \leq 1$, along with accompanying examples. Finally, the discussion and conclusion are presented in Section 4.

Throughout this paper, the following notations are used: $\mu = E(X)$, $\sigma^2 = Var(X)$, $\mu_t = E|X|^t$ for $t \in \mathbb{R}^+$, and $p_m = P(X = m)$ for $m \in \mathbb{N}$. Let δ be a real number such that $0 < \delta \leq 1$.

2 Compound Binomial Collective Risk Model

Let B be a binomial random variable with parameter $n \in \mathbb{N}$ and $p \in (0, 1)$. Define a compound binomial collective risk model S_B by:

$$S_B = \sum_{i=1}^{B} X_i.$$

In this model, the number of claims is assumed to follow a binomial distribution. Assume that $B, X, X_1, X_2, \ldots$ are independent. We obtain that $E(S_B) = E(B)E(X) = np\mu$ and $Var(S_B) = E(B)Var(X) + Var(B)E^2(X) = np\sigma^2 + np(1-p)\mu^2 = np\gamma_p$, where $\gamma_p = \mu_2 - p\mu^2$. Set,

$$\alpha_X = 2 \sum_{m \in \operatorname{Im} X} p_m p_{m+1}.$$

In 2023, Kongjiw et al. [8] provided a bound in the local limit theorem with the rate of convergence $\mathcal{O}\left(\frac{1}{n^{(1+\delta)/2}}\right)$. The following is their result.

Theorem 2.1 ([8]). *Assume that $\mu_{2+\delta} < \infty$. For any $n \in \mathbb{N}$ and $p \in (0, 1)$, we have*

$$\sup_{k \in \mathbb{N}} \left| P(S_B = k) - \frac{1}{\sqrt{2\pi np\gamma_p}} e^{-\frac{(k-np\mu)^2}{2np\gamma_p}} \right| \leq \frac{C_1}{n^{(1+\delta)/2}},$$

where

$$C_1 = \frac{\pi}{2[2(1-p)pp_1 + p^2\alpha_X]T_1} + \frac{3 \cdot 2^{3+\delta}\mu_{2+\delta}}{\pi p^{(1+\delta)/2}\gamma_p^{(3+\delta)/2}} + \frac{1}{\pi p\gamma_p T_1},$$
$$T_1 = \left(\frac{\gamma_p}{24\mu_{2+\delta}}\right)^{1/\delta}.$$

The aim of this work is to refine the constant C_1 in Theorem 2.1 by adjusting the term T_1 and expressing some bounds in exponential forms. To underscore the precision of the bound, we present a remark and an illustrative example later. The obtained result is as follows.

Theorem 2.2. *Assume that $\mu_{2+\delta} < \infty$. For any $n \in \mathbb{N}$ and $p \in (0,1)$, we have*

$$\sup_{k\in\mathbb{N}} \left| P(S_B = k) - \frac{1}{\sqrt{2\pi np\gamma_p}} e^{-\frac{(k-np\mu)^2}{2np\gamma_p}} \right| \le \frac{C_2}{n^{(1+\delta)/2}},$$

where

$$C_2 = \frac{\pi e^{-n[2(1-p)pp_1+p^2\alpha_X]T_2^2/\pi^2}}{2[2(1-p)pp_1 + p^2\alpha_X]T_2 n^{(1-\delta)/2}} + \frac{3 \cdot (8/3)^{(3+\delta)/2}\mu_{2+\delta}}{\pi p^{(1+\delta)/2}\gamma_p^{(3+\delta)/2}} + \frac{e^{-np\gamma_p T_2^2/2}}{\pi p\gamma_p T_2 n^{(1-\delta)/2}},$$
$$T_2 = \left(\frac{\gamma_p}{48\mu_{2+\delta}}\right)^{1/\delta}.$$

Proof. The fundamental concept of the proof involves rewriting the random sum S_B into a non-random sum, drawing inspiration from the work of Korolev and Dorofeeva [9, p.48]. Let $Y_1, Y_2, Y_3, \ldots$ be independent and identically distributed Bernoulli random variables with parameter p and $Y \overset{d}{=} Y_1$. Assume that $X_1, X_2, X_3, \ldots, Y_1, Y_2, Y_3, \ldots$ are independent. In 2017, Korolev and Dorofeeva [9, p.48] showed that a binomial random sum S_B can be represented in a non-random sum as follows:

$$S_B \overset{d}{=} X_1Y_1 + X_2Y_2 + X_3Y_3 + \ldots + X_nY_n. \tag{1}$$

Following this, Kongjiw et al. [8] adopted an idea from [9] and utilized the triangle inequality to establish that, for any real number T such that $0 < T \le \pi$, the following holds:

$$\left| P(S_B = k) - \frac{1}{2\pi np\gamma_p} e^{-\frac{(k-np\mu)^2}{2np\gamma_p}} \right| \le A_1 + A_2 + A_3, \tag{2}$$

where

$$A_1 = \frac{1}{2\pi}\int_{T<|t|\le\pi} |\varphi_{S_B}(t)|\, dt, \tag{3}$$
$$A_2 = \frac{1}{2\pi}\int_{|t|\le T} \left|\varphi_{S_B}(t) - e^{inp\mu t - np\gamma_p t^2/2}\right| dt, \tag{4}$$
$$A_3 = \frac{1}{2\pi}\int_{|t|>T} e^{-np\gamma_p t^2/2} dt, \tag{5}$$

and φ_{S_B} is the characteristic function of S_B. Additionally, they showed that for any real number T such that $0 < T \le \pi$, the following hold [8, p.3]:

$$A_1 \le \frac{\pi}{2n[2(1-p)pp_1 + p^2\alpha_X]T} e^{-n[2(1-p)pp_1+p^2\alpha_X]T^2/\pi^2}, \tag{6}$$
$$A_3 \le \frac{1}{\pi np\gamma_p T} e^{-np\gamma_p T^2/2}. \tag{7}$$

In this study, we define T in (3)–(5) as:

$$T_2 = \left(\frac{\gamma_p}{48\mu_{2+\delta}}\right)^{1/\delta}.$$

It is true that $0 < T_2 \leq 1 < \pi$ since $\gamma_p = \mu_2 - p\mu^2 \leq \mu_2 \leq \mu_{2+\delta}$. From T_2 and (6)–(7), we obtain that,

$$A_1 \leq \frac{\pi}{2n[2(1-p)pp_1 + p^2\alpha_X]T_2} e^{-n[2(1-p)pp_1+p^2\alpha_X]T_2^2/\pi^2}, \tag{8}$$

$$A_3 \leq \frac{1}{\pi np\gamma_p T_2} e^{-np\gamma_p T_2^2/2}. \tag{9}$$

To find a bound for A_2, we first notice that Sunklodas [16, p.358] and Kongjiw et al. [8, p.4] have shown that,

$$\left|\varphi_{S_B}(t) - e^{inp\mu t - np\gamma_p t^2/2}\right| < 6\mu_{2+\delta} np|t|^{2+\delta} e^{-\epsilon np\gamma_p t^2},$$

for $|t| \leq \left(\frac{2^\delta(1+\delta)(2+\delta)(1-2\epsilon)\gamma_p}{8(3+\delta)\mu_{2+\delta}}\right)^{1/\delta}$ and $0 < \epsilon < \frac{1}{2}$. In this work, we choose $\epsilon = \frac{3}{8}$ and obtain that,

$$\left|\varphi_{S_B}(t) - e^{inp\mu t - np\gamma_p t^2/2}\right| < 6\mu_{2+\delta} np|t|^{2+\delta} e^{-3np\gamma_p t^2/8}, \tag{10}$$

for $|t| \leq \left(\frac{2^\delta(1+\delta)(2+\delta)\gamma_p}{32(3+\delta)\mu_{2+\delta}}\right)^{1/\delta}$. By the fact that $\frac{3+\delta}{2^\delta(1+\delta)(2+\delta)} < \frac{3}{2}$ [8, p.4], our choosed T_2 is applicable with the condition of (10). This implies that, for $|t| \leq T_2$,

$$\begin{aligned} A_2 &= \frac{1}{2\pi}\int_{|t|\leq T_2} \left|\varphi_{S_B}(t) - e^{inp\mu t - np\gamma_p t^2/2}\right| dt \\ &\leq \frac{3}{\pi}\mu_{2+\delta} np \int_{|t|\leq T_2} |t|^{2+\delta} e^{-3np\gamma_p t^2/8} dt. \end{aligned}$$

Utilizing the following integral formula:

$$\int_0^\infty x^{a-1} e^{-bx^c} dx = \frac{1}{c} b^{-a/c}\Gamma\left(\frac{a}{c}\right), \tag{11}$$

for $a, b, c \in \mathbb{R}^+$ ([4], p.370), we obtain that,

$$\begin{aligned} A_2 &\leq \frac{3\cdot(8/3)^{(3+\delta)/2}\mu_{2+\delta}}{\pi(np)^{(1+\delta)/2}\gamma_p^{(3+\delta)/2}}\Gamma\left(\frac{3+\delta}{2}\right) \\ &\leq \frac{3\cdot(8/3)^{(3+\delta)/2}\mu_{2+\delta}}{\pi(np)^{(1+\delta)/2}\gamma_p^{(3+\delta)/2}}. \end{aligned} \tag{12}$$

Combining (8)–(9) and (12), we have the theorem as desired. □

Remark 2.1. *The finding presented in Theorem 2.2 refines the constant in the bound established in Theorem 2.1. To facilitate a comprehensive comparison of the bounds, consider*

$$\frac{3\cdot(8/3)^{(3+\delta)/2}\mu_{2+\delta}}{\pi p^{(1+\delta)/2}\gamma_p^{(3+\delta)/2}} = \frac{3\cdot 2^{3+\delta}\mu_{2+\delta}}{\pi p^{(1+\delta)/2}\gamma_p^{(3+\delta)/2}}\cdot\left(\frac{2}{3}\right)^{(3+\delta)/2} < \frac{3\cdot 2^{3+\delta}\mu_{2+\delta}}{\pi p^{(1+\delta)/2}\gamma_p^{(3+\delta)/2}}.$$

Additionally, for sufficiently large n such that,

$$2^{1/\delta} \leq n^{(1-\delta)/2} \min\left\{ e^{n[2(1-p)pp_1+p^2\alpha_X]T_2^2/\pi^2}, e^{np\gamma_p T_2^2/2} \right\}, \tag{13}$$

our remaining error bounds are sharper than those in Theorem 2.1 since,

$$\begin{aligned}
&\frac{\pi}{2[2(1-p)pp_1+p^2\alpha_X]T_2 n^{(1-\delta)/2}} e^{-n[2(1-p)pp_1+p^2\alpha_X]T_2^2/\pi^2} \\
&\quad = \frac{\pi}{2[2(1-p)pp_1+p^2\alpha_X]T_1 n^{(1-\delta)/2}} \cdot 2^{1/\delta} e^{-n[2(1-p)pp_1+p^2\alpha_X]T_2^2/\pi^2} \\
&\quad \leq \frac{\pi}{2[2(1-p)pp_1+p^2\alpha_X]T_1},
\end{aligned}$$

and

$$\frac{1}{\pi p\gamma_p T_2 n^{(1-\delta)/2}} e^{-np\gamma_p T_2^2/2} = \frac{1}{\pi p\gamma_p T_1} \frac{2^{1/\delta} e^{-np\gamma_p T_2^2/2}}{n^{(1-\delta)/2}} \leq \frac{1}{\pi p\gamma_p T_1}.$$

Next, we propose a graphical illustration of the error bounds to provide a clearer comparison between Theorems 2.1 and 2.2. The presented random variable has an infinite third moment and a finite $(2+\delta)$-th moment, where $0 < \delta < 1$. In addition, it serves as a suitable example in insurance, as it reflects heavy-tailed behavior.

Example 2.1. *Let $X, X_1, X_2, X_3, \ldots$ be independent and identically distributed random variables defined by:*

$$P(X = k) = \frac{90}{\pi^4 k^4}, \quad \textit{for } k = 1, 2, 3, \ldots. \tag{14}$$

Notice that X has an infinite third moment and a finite $(2+\delta)$-th moment, where $0 < \delta < 1$. Additionally, it can be shown that,

$$\mu = \frac{90}{\pi^4}\zeta(3), \quad \mu_2 = \frac{15}{\pi^2}, \quad \mu_{2+\delta} = \frac{90}{\pi^4}\zeta(2-\delta), \quad \textit{and} \quad \alpha_X = \frac{16200}{\pi^8}\left(-35 + \frac{10\pi^2}{3} + \frac{\pi^4}{45}\right),$$

where ζ is the Riemann zeta function defined by $\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$ for complex number x such that $\Re(x) > 1$.

Using this information, setting the parameter p of the binomial random variable B to 0.4, and $\delta = 0.6$ yields Figure 1.

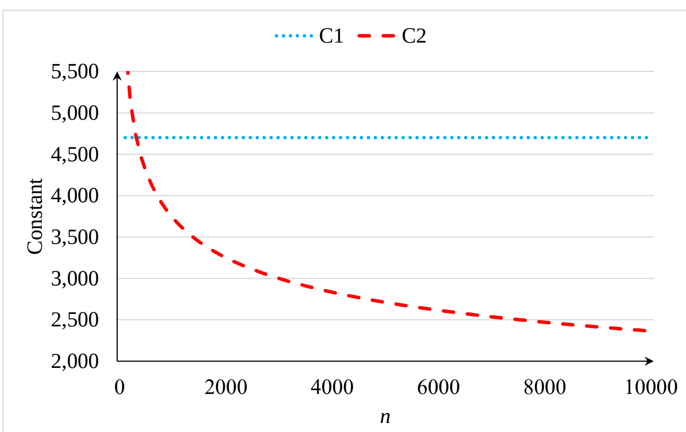

Figure 1: Constants C_1 and C_2 of error bounds for collective risk model in Theorems 2.1 and 2.2 when $\delta = 0.6$ and $p = 0.4$.

It is worth noting that Theorem 2.2 provides an error bound for a sequence of random variables with a finite third moment when $\delta = 1$. However, to improve the sharpness of the result, we propose a refinement rather than directly applying Theorem 2.2 with $\delta = 1$. Moreover, the result improves the bound given in [7].

In 2023, Kongjiw et al. [7] established an error bound for the collective risk model under the assumption that $\mu_3 < \infty$ as shown.

Theorem 2.3 ([7]). *Assume that $\mu_3 < \infty$. For any $n \in \mathbb{N}$ and $p \in (0,1)$, we have*

$$\sup_{k\in\mathbb{N}}\left|P(S_B = k) - \frac{1}{\sqrt{2\pi np\gamma_p}}e^{-\frac{(k-np\mu)^2}{2np\gamma_p}}\right| \le \frac{C_3\mu_3}{n},$$

where $C_3 = \dfrac{4.89}{[2(1-p)pp_1 + p^2\alpha_X]\gamma_p} + \dfrac{2.97}{p\gamma_p^2}$.

The next theorem shows our refined outcome.

Theorem 2.4. *Suppose that $\mu_3 < \infty$. For any $n \in \mathbb{N}$ and $p \in (0,1)$, we have*

$$\sup_{k\in\mathbb{N}}\left|P(S_B = k) - \frac{1}{\sqrt{2\pi np\gamma_p}}e^{-\frac{(k-np\mu)^2}{2np\gamma_p}}\right| \le \frac{C_4\mu_3}{n},$$

where $C_4 = \dfrac{9.78}{[2(1-p)pp_1 + p^2\alpha_X]\gamma_p}e^{-\left(\frac{9}{56}\right)^2\frac{n[2(1-p)pp_1+p^2\alpha_X]\gamma_p^2}{\pi^2\mu_3^2}} + \dfrac{0.89}{p\gamma_p^2} + \dfrac{1.99}{p\gamma_p^2}e^{-\frac{1}{2}\left(\frac{9}{56}\right)^2\frac{np\gamma_p^3}{\mu_3^2}}$.

Proof. From (3)–(5), we modify the formula for T to be,

$$T_3 = \frac{9\gamma_p}{56\mu_3}.$$

Based on the revised formula for T_3 and utilizing (6)–(7), we derive alternative bounds for A_1 and A_3 under the condition that $\mu_3 < \infty$ as follows:

$$A_1 \le \frac{\pi}{2n[2(1-p)pp_1 + p^2\alpha_X]T_3}e^{-n[2(1-p)pp_1+p^2\alpha_X]T_3^2/\pi^2} \tag{15}$$

$$\le \frac{9.78\mu_3}{n[2(1-p)pp_1 + p^2\alpha_X]\gamma_p}e^{-\left(\frac{9}{56}\right)^2\frac{n[2(1-p)pp_1+p^2\alpha_X]\gamma_p^2}{\pi^2\mu_3^2}}, \tag{16}$$

and

$$A_3 \le \frac{1}{\pi np\gamma_pT_3}e^{-np\gamma_pT_3^2/2} \tag{17}$$

$$= \frac{1.99\mu_3}{np\gamma_p^2}e^{-\frac{1}{2}\left(\frac{9}{56}\right)^2\frac{np\gamma_p^3}{\mu_3^2}}. \tag{18}$$

To verify the bound for A_2, Kongjiw et al. [7, p.120] showed that,

$$\left|\varphi_{S_B}(t) - e^{inp\mu t - np\gamma_p t^2/2}\right| \le \frac{7}{9}\mu_3 np|t|^3 e^{-\epsilon np\gamma_p t^2},$$

for $|t| \leq \frac{9\gamma_p}{7\mu_3}\left(\frac{1}{2} - \epsilon\right)$ and $0 < \epsilon < \frac{1}{2}$. Choosing $\epsilon = \frac{3}{8}$, we obtain that,

$$\left|\varphi_{S_B}(t) - e^{inp\mu t - np\gamma_p t^2/2}\right| \leq \frac{7}{9}\mu_3 np|t|^3 e^{-3np\gamma_p t^2/8},$$

for $|t| \leq \frac{9\gamma_p}{56\mu_3} = T_3$. From this fact and (11), we have

$$\begin{aligned} A_2 &= \frac{1}{2\pi}\int_{|t|\leq T_3}\left|\varphi_{S_B}(t) - e^{inp\mu t - np\gamma_p t^2/2}\right| dt \\ &\leq \frac{7}{9\pi} np\mu_3 \int_0^\infty |t|^3 e^{-3np\gamma_p t^2/8} dt \\ &\leq \frac{0.89\mu_3}{np\gamma_p^2}\Gamma(2) \\ &= \frac{0.89\mu_3}{np\gamma_p^2}. \end{aligned} \tag{19}$$

Combining (16)–(19), we have the theorem as required. □

Remark 2.2. *To clarify the sharpness of our theorem under the condition $\mu_3 < \infty$, we need to compare the constant terms,*

$$\frac{4.89}{[2(1-p)pp_1 + p^2\alpha_X]\gamma_p} + \frac{2.98}{p\gamma_p^2},$$

in Theorem 2.3 with the constant terms,

$$\frac{9.78}{[2(1-p)pp_1 + p^2\alpha_X]\gamma_p} e^{-\left(\frac{9}{56}\right)^2 \frac{n[2(1-p)pp_1 + p^2\alpha_X]\gamma_p^2}{\pi^2\mu_3^2}} + \frac{0.89}{p\gamma_p^2} + \frac{1.99}{p\gamma_p^2} e^{-\frac{1}{2}\left(\frac{9}{56}\right)^2 \frac{np\gamma_p^3}{\mu_3^2}},$$

in Theorem 2.4. It is clear that,

$$1.99 e^{-\frac{1}{2}\left(\frac{9}{56}\right)^2 \frac{np\gamma_p^3}{\mu_3^2}} < 2.08.$$

Additionally, we notice that for sufficient large n such that,

$$n[2(1-p)pp_1 + p^2\alpha_X]\gamma_p^2 > 264.86\mu_3^2, \tag{20}$$

the following inequality holds:

$$9.78 e^{-\left(\frac{9}{56}\right)^2 \frac{n[2(1-p)pp_1 + p^2\alpha_X]\gamma_p^2}{\pi^2\mu_3^2}} \leq 4.89.$$

The next example demonstrates the sharpness of constant C_4 compared with C_3.

Example 2.2. *Let $X, X_1, X_2, \ldots$ be independent random variables having geometric distribution with parameter 0.8 defined by:*

$$P(X = m) = 0.8(0.2)^{m-1}, \text{ for } m = 1, 2, 3 \ldots. \tag{21}$$

One can compute the following quantities:

$$\mu = \frac{5}{4}, \quad \mu_2 = \frac{15}{8}, \quad \gamma_p = \frac{15}{8} - \frac{25}{16}p, \quad \mu_3 = \frac{115}{32}, \quad \text{and} \quad \alpha_X = \frac{4}{15}. \tag{22}$$

Let the parameter p of the binomial random variable B be 0.1. From (21)–(22), we obtain Figure 2.

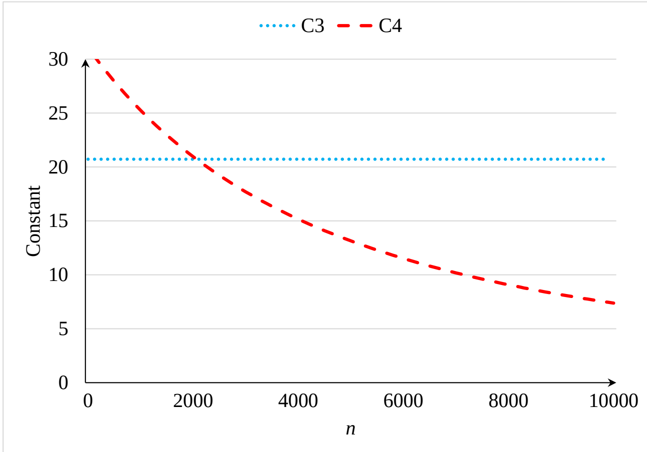

Figure 2: Constants C_3 and C_4 of error bounds for collective risk model in Theorems 2.3 and 2.4 when $p = 0.1$.

3 Compound Poisson Collective Risk Model

In this section, our focus lies on a compound Poisson collective risk model, in which the amount of claims follows a Poisson random variable. Let P_λ be a Poisson random variable with parameter $\lambda > 0$. Define a compound Poisson collective risk model S_{P_λ} by:

$$S_{P_\lambda} = \sum_{i=1}^{P_\lambda} X_i.$$

Assume that $P_\lambda, X, X_1, X_2, \ldots$ are independent. We obtain that $E(S_{P_\lambda}) = E(P_\lambda)E(X) = \lambda\mu$ and $Var(S_{P_\lambda}) = E(P_\lambda)Var(X) + Var(P_\lambda)E^2(X) = \lambda(\mu_2 - \mu^2) + \lambda\mu^2 = \lambda\mu_2$. In 2023, Kongjiw et al. [8] also provided a bound for approximating the compound Poisson collective risk model as shown in the following theorem.

Theorem 3.1 ([8])**.** *Assume that $\mu_{2+\delta} < \infty$. For any $\lambda > 0$, we have*

$$\sup_{k\in\mathbb{N}} \left| P(S_{P_\lambda} = k) - \frac{1}{\sqrt{2\pi\lambda\mu_2}} e^{-\frac{(k-\lambda\mu)^2}{2\lambda\mu_2}} \right| \leq \frac{D_1}{\lambda} + \frac{D_2}{\lambda^{(1+\delta)/2}},$$

where

$$D_1 = \frac{\pi\left(24\mu_{2+\delta}\right)^{1/\delta}}{4p_1\mu_2^{1/\delta}} + \frac{\left(24\mu_{2+\delta}\right)^{1/\delta}}{\pi\mu_2^{(1+\delta)/\delta}},$$

$$D_2 = \frac{3\cdot 2^{3+\delta}\mu_{2+\delta}}{\pi\mu_2^{(3+\delta)/2}}.$$

In this work, we improve the bound in Theorem 3.1. The following is our result.

Theorem 3.2. *Assume that $\mu_{2+\delta} < \infty$. For any $\lambda > 0$, we have*

$$\sup_{k\in\mathbb{N}} \left| P(S_{P_\lambda} = k) - \frac{1}{\sqrt{2\pi\lambda\mu_2}} e^{-\frac{(k-\lambda\mu)^2}{2\lambda\mu_2}} \right| \leq \frac{D_3}{\lambda} + \frac{D_4}{\lambda^{(1+\delta)/2}},$$

where

$$D_3 = \frac{\pi(48\mu_{2+\delta})^{1/\delta}}{4p_1\mu_2^{1/\delta}} e^{-\frac{2\lambda p_1 \mu_2^{2/\delta}}{\pi^2(48\mu_{2+\delta})^{2/\delta}}} + \frac{(48\mu_{2+\delta})^{1/\delta}}{\pi\mu_2^{(1+\delta)/\delta}} e^{-\frac{\lambda\mu_2^{(2+\delta)/\delta}}{2(48\mu_{2+\delta})^{2/\delta}}},$$
$$D_4 = \frac{3\cdot(8/3)^{(3+\delta)/2}\mu_{2+\delta}}{\pi\mu_2^{(3+\delta)/2}}.$$

Proof. To obtain a result about the compound Poisson collective risk model motivated by [8], we essentially need the result from the compound binomial collective risk model and utilizing a triangle inequality. Let n be a positive integer such that $0 < \frac{\lambda}{n} < 1$ and P_n a binomial random variable with parameters n and $\frac{\lambda}{n}$. Assume that $P_n, X_1, X_2, X_3, \ldots$ are independent. Kongjiw et al. [8] utilized the triangle inequality to prove that,

$$\left| P(S_{P_\lambda} = k) - \frac{1}{\sqrt{2\pi\lambda\mu_2}} e^{-\frac{(k-\lambda\mu)^2}{2\lambda\mu_2}} \right| \le B_1 + B_2 + B_3,$$

where

$$B_1 = \left| P(S_{P_\lambda} = k) - P(S_{P_n} = k) \right|,$$
$$B_2 = \left| P(S_{P_n} = k) - \frac{1}{\sqrt{2\pi\lambda\gamma_{\frac{\lambda}{n}}}} e^{-\frac{(k-\lambda\mu)^2}{2\lambda\gamma_{\frac{\lambda}{n}}}} \right|,$$
$$B_3 = \left| \frac{1}{\sqrt{2\pi\lambda\gamma_{\frac{\lambda}{n}}}} e^{-\frac{(k-\lambda\mu)^2}{2\lambda\gamma_{\frac{\lambda}{n}}}} - \frac{1}{\sqrt{2\pi\lambda\mu_2}} e^{-\frac{(k-\lambda\mu)^2}{2\lambda\mu_2}} \right|.$$

In addition, they showed that,

$$\lim_{n\to\infty} B_1 = \lim_{n\to\infty} B_3 = 0.$$

By the fact that $\lim_{n\to\infty} \gamma_{\frac{\lambda}{n}} = \mu_2$, $\lim_{n\to\infty} np[2(1-p)p_1 + p\alpha_X] = 2\lambda p_1$, and utilizing Theorem 2.2, we obtain a refined version for compound Poisson collective risk model as follows:

$$\begin{aligned}
&\left| P(S_{P_\lambda} = k) - \frac{1}{\sqrt{2\pi\lambda\mu_2}} e^{-\frac{(k-\lambda\mu)^2}{2\lambda\mu_2}} \right| \\
&\le \frac{\pi(48\mu_{2+\delta})^{1/\delta}}{4p_1\lambda\mu_2^{1/\delta}} e^{-\frac{2\lambda p_1 \mu_2^{2/\delta}}{\pi^2(48\mu_{2+\delta})^{2/\delta}}} + \frac{3\cdot(8/3)^{(3+\delta)/2}\mu_{2+\delta}}{\pi\lambda^{(1+\delta)/2}\mu_2^{(3+\delta)/2}} + \frac{(48\mu_{2+\delta})^{1/\delta}}{\pi\lambda\mu_2^{(1+\delta)/\delta}} e^{-\frac{\lambda\mu_2^{(2+\delta)/\delta}}{2(48\mu_{2+\delta})^{2/\delta}}} \\
&= \frac{D_3}{\lambda} + \frac{D_4}{\lambda^{(1+\delta)/2}},
\end{aligned}$$

where

$$D_3 = \frac{\pi(48\mu_{2+\delta})^{1/\delta}}{4p_1\mu_2^{1/\delta}} e^{-\frac{2\lambda p_1 \mu_2^{2/\delta}}{\pi^2(48\mu_{2+\delta})^{2/\delta}}} + \frac{(48\mu_{2+\delta})^{1/\delta}}{\pi\mu_2^{(1+\delta)/\delta}} e^{-\frac{\lambda\mu_2^{(2+\delta)/\delta}}{2(48\mu_{2+\delta})^{2/\delta}}},$$
$$D_4 = \frac{3\cdot(8/3)^{(3+\delta)/2}\mu_{2+\delta}}{\pi\mu_2^{(3+\delta)/2}}.$$

□

Remark 3.1. *The constant of the error bound in Theorem 3.2 is sharper than that in Theorem 3.1 as demonstrated by the following considerations:*

$$D_4 = \frac{3 \cdot (8/3)^{(3+\delta)/2} \mu_{2+\delta}}{\pi \mu_2^{(3+\delta)/2}} = D_2 \cdot \left(\frac{2}{3}\right)^{(3+\delta)/2} < D_2,$$

and for sufficient large λ such that,

$$2^{1/\delta} \leq \min \left\{ e^{\frac{2\lambda p_1 \mu_2^{2/\delta}}{\pi^2 (48\mu_{2+\delta})^{2/\delta}}}, e^{\frac{\lambda \mu_2^{(2+\delta)/\delta}}{2(48\mu_{2+\delta})^{2/\delta}}} \right\}, \tag{23}$$

we have

$$\begin{aligned} D_3 &= \frac{\pi (48\mu_{2+\delta})^{1/\delta}}{4p_1 \mu_2^{1/\delta}} e^{-\frac{2\lambda p_1 \mu_2^{2/\delta}}{\pi^2 (48\mu_{2+\delta})^{2/\delta}}} + \frac{(48\mu_{2+\delta})^{1/\delta}}{\pi \mu_2^{(1+\delta)/\delta}} e^{-\frac{\lambda \mu_2^{(2+\delta)/\delta}}{2(48\mu_{2+\delta})^{2/\delta}}} \\ &= \frac{\pi \left(24\mu_{2+\delta}\right)^{1/\delta}}{4p_1 \mu_2^{1/\delta}} \cdot 2^{1/\delta} e^{-\frac{2\lambda p_1 \mu_2^{2/\delta}}{\pi^2 (48\mu_{2+\delta})^{2/\delta}}} + \frac{\left(24\mu_{2+\delta}\right)^{1/\delta}}{\pi \mu_2^{(1+\delta)/\delta}} \cdot 2^{1/\delta} e^{-\frac{\lambda \mu_2^{(2+\delta)/\delta}}{2(48\mu_{2+\delta})^{2/\delta}}} \\ &\leq D_1. \end{aligned}$$

Example 3.1. *Let $X, X_1, X_2, X_3, \ldots$ be independent and identically distributed random variables defined as in (14). The error bounds in Theorems 3.1 and 3.2 are computed for the case where $\delta = 0.5$. Observe that, $D_4 = \left(\frac{2}{3}\right)^{3.5/2} D_2 \leq 0.492 D_2$. This shows that the constant D_4 refines the constant D_2. Moreover, by varying the parameter λ of the Poisson random variable P_λ, we establish a comparison of the constants D_1 and D_3 in Figure 3.*

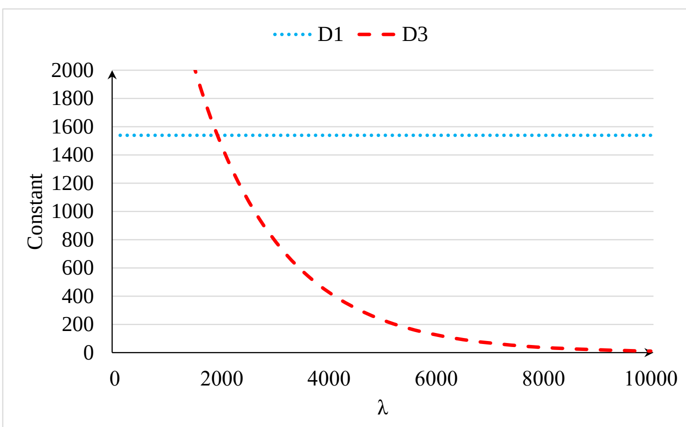

Figure 3: Constants D_1 and D_3 of error bounds for collective risk model in Theorems 3.1 and 3.2.

Next, we provide an error bound for a Poisson collective risk model under the assumption that $\mu_3 < \infty$. Before proceeding the proof, we first examine the recent result established by Kongjiw et al. [7] as presented below.

Theorem 3.3 ([7]). *Assume that $\mu_3 < \infty$. For any $\lambda > 0$, we have*

$$\sup_{k \in \mathbb{N}} \left| P(S_{P_\lambda} = k) - \frac{1}{\sqrt{2\pi\lambda\mu_2}} e^{-\frac{(k-\lambda\mu)^2}{2\lambda\mu_2}} \right| \leq \left(\frac{2.45}{p_1 \mu_2} + \frac{2.97}{\mu_2^2} \right) \frac{\mu_3}{\lambda}.$$

We provide the refined version in the following theorem.

Theorem 3.4. *Assume that $\mu_3 < \infty$. For any $\lambda > 0$, we have*

$$\sup_{k\in\mathbb{N}} \left| P(S_{P_\lambda} = k) - \frac{1}{\sqrt{2\pi\lambda\mu_2}} e^{-\frac{(k-\lambda\mu)^2}{2\lambda\mu_2}} \right| \le \left(\frac{4.89}{p_1\mu_2} e^{-\left(\frac{9}{56}\right)^2 \frac{2\lambda p_1 \mu_2^2}{\pi^2 \mu_3^2}} + \frac{0.89}{\mu_2^2} + \frac{1.99}{\mu_2^2} e^{-\frac{1}{2}\left(\frac{9}{56}\right)^2 \frac{\lambda\mu_2^3}{\mu_3^2}} \right) \frac{\mu_3}{\lambda}.$$

Proof. Following the argument in Theorem 3.2 and the result in Theorem 2.4, under the condition that $\mu_3 < \infty$, we have

$$\begin{aligned} &\left| P(S_{P_\lambda} = k) - \frac{1}{\sqrt{2\pi\lambda\mu_2}} e^{-\frac{(k-\lambda\mu)^2}{2\lambda\mu_2}} \right| \\ &\le \frac{4.89\mu_3}{p_1\mu_2\lambda} e^{-\left(\frac{9}{56}\right)^2 \frac{2\lambda p_1 \mu_2^2}{\pi^2 \mu_3^2}} + \frac{0.89\mu_3}{\mu_2^2\lambda} + \frac{1.99\mu_3}{\mu_2^2\lambda} e^{-\frac{1}{2}\left(\frac{9}{56}\right)^2 \frac{\lambda\mu_2^3}{\mu_3^2}}. \end{aligned}$$

□

Remark 3.2. *To compare the sharpness between the constants in Theorems 3.3 and 3.4, it is clear that,*

$$1.99e^{-\frac{1}{2}\left(\frac{9}{56}\right)^2 \frac{\lambda\mu_2^3}{\mu_3^2}} \le 2.08,$$

and for sufficiently large λ such that,

$$132.04 \le \frac{\lambda p_1 \mu_2^2}{\mu_3^2}, \tag{24}$$

we have that,

$$4.89e^{-\left(\frac{9}{56}\right)^2 \frac{2\lambda p_1 \mu_2^2}{\pi^2 \mu_3^2}} < 2.45.$$

Example 3.2. *Let $X, X_1, X_2, X_3, \ldots$ be independent and identically distributed random variables defined as in (21). For convenience, denote,*

$$\begin{aligned} D_5 &= \frac{2.45}{p_1\mu_2}, \\ D_6 &= \frac{2.97}{\mu_2^2}, \\ D_7 &= \frac{4.89}{p_1\mu_2} e^{-\left(\frac{9}{56}\right)^2 \frac{2\lambda p_1 \mu_2^2}{\pi^2 \mu_3^2}}, \\ D_8 &= \frac{0.89}{\mu_2^2} + \frac{1.99}{\mu_2^2} e^{-\frac{1}{2}\left(\frac{9}{56}\right)^2 \frac{\lambda\mu_2^3}{\mu_3^2}}. \end{aligned}$$

From the moments in (22), the graphical comparisons of all bounds are shown in Figures 4 and 5. It is evident that the error bounds D_7 and D_8 steadily decrease as λ increases.

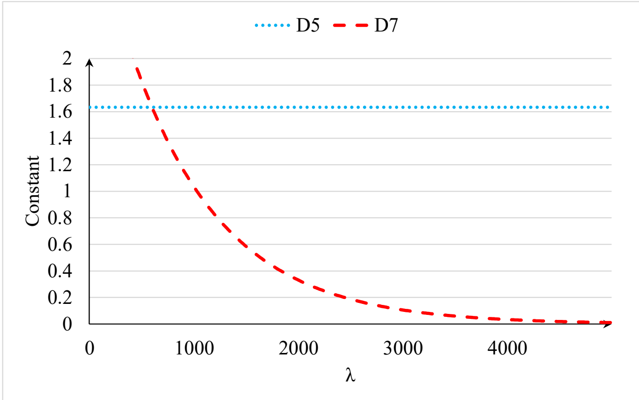

Figure 4: Constants D_5 and D_7 of error bounds for collective risk model in Theorems 3.3 and 3.4.

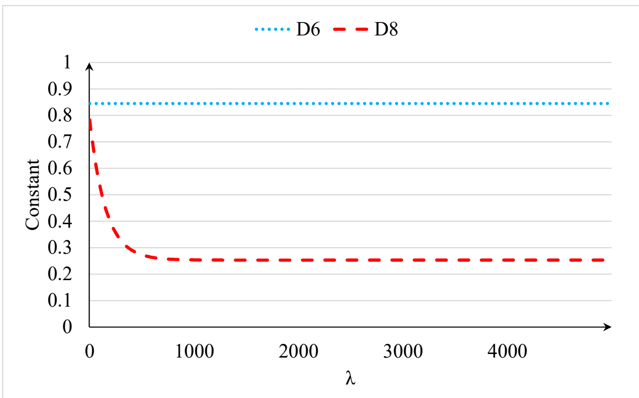

Figure 5: Constants D_6 and D_8 of error bounds for collective risk model in Theorems 3.3 and 3.4.

4 Discussion and Conclusion

The obtained results in the compound binomial collective risk model are sharper than those in [8], where the parameter n of the binomial random variable B exceeds a certain threshold, and the results continue to decrease as n increases as explicitly demonstrated in Figures 1–2. Similarly, the error bounds for the compound Poisson collective risk model improve the constants of the bounds in [8], when the parameter λ of the Poisson random variable exceeds a certain threshold, and the bounds decrease as λ increases. The demonstrations are presented in Figures 3–5. Additionally, the constants in the bounds are sharp as δ approaches 1, indicating that the approximated collective risk model exhibits the finiteness of the third moment. However, even when the third moment of the random variables do not exist, the error bounds in the collective risk model remain valid, albeit with less precise constants. For small values of δ approaching zero, the constants in the bounds presented in Theorems 2.1–2.2 and Theorems 3.1–3.2 increase in both the compound binomial and compound Poisson collective risk models. The effect of δ on the bounds arises from the fact that $\gamma_p = \mu_2 - p\mu^2 < \mu_2 < \mu_{2+\delta}$, which implies that both $T_1 = \left(\dfrac{\gamma_p}{24\mu_{2+\delta}}\right)^{1/\delta}$

and $T_2 = \left(\dfrac{\gamma_p}{48\mu_{2+\delta}}\right)^{1/\delta}$ tend to zero as δ approaches zero in the compound binomial model. Additionally, in the compound Poisson model, the term $\left(\dfrac{\mu_{2+\delta}}{\mu_2}\right)^{1/\delta}$ increases as δ approaches zero, contributing to the growth of the bounds.

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References

[1] A. D. Barbour, P. Braunsteins & N. Ross (2021). Local limit theorems for occupancy models. *Random Structures & Algorithms*, *58*(1), 3–33. https://doi.org/10.1002/rsa.20967.

[2] A. C. Berry (1941). The accuracy of the Gaussian approximation to the sum of independent variates. *Transactions of the American Mathematical Society*, *49*(1), 122–136. https://doi.org/10.1090/S0002-9947-1941-0003498-3.

[3] C. G. Esseen (1945). Fourier analysis of distribution functions. A mathematical study of the Laplace-Gaussian law. *Acta Mathematics*, *77*, 1–125. https://doi.org/10.1007/BF02392223.

[4] A. Jeffrey, D. Zwillinger, I. S. Gradshteyn & I. Ryzhik (2007). 3-4-Definite integrals of elementary functions. In *Table of Integrals, Series, and Products*, pp. 247–617. Academic Press, California, USA. https://doi.org/10.1016/B978-012294757-5/50010-6.

[5] R. Kaas, M. Goovaerts, J. Dhaene & M. Denuit (2008). *Modern Actuarial Risk Theory: Using R*. Springer, Berlin, Heidelberg. https://doi.org/10.1007/978-3-540-70998-5.

[6] P. Kammoo, K. Laipaporn & K. Neammanee (2023). Local limit theorems without assuming finite third moment. *Journal of Inequalities and Applications*, *2023*(1), Article ID: 21. https://doi.org/10.1186/s13660-023-02928-y.

[7] H. Kongjiw, P. Rattanawong & K. Neammanee (2023). Local limit theorem for collective risk models assuming finite absolute third moment claims. In *Conference Proceedings: The 27th Annual Meeting in Mathematics (AMM2023) and International Conference in Number Theory and Applications (ICNA2023)*, volume 1 pp. 117–126. Bangkok, Thailand.

[8] H. Kongjiw, P. Rattanawong & K. Neammanee (2023). Local limit theorems for collective risk models. *Statistics & Probability Letters*, *201*, Article ID: 109867. https://doi.org/10.1016/j.spl.2023.109867.

[9] V. Korolev & A. Dorofeeva (2017). Bounds of the accuracy of the normal approximation to the distributions of random sums under relaxed moment conditions. *Lithuanian Mathematical Journal*, *57*(1), 38–58. https://doi.org/10.1007/s10986-017-9342-7.

[10] T. Manoharan, J. Arasan, H. Midi & M. B. Adam (2015). A coverage probability on the parameters of the log-normal distribution in the presence of left-truncated and right-censored survival data. *Malaysian Journal of Mathematical Sciences*, *9*(1), 127–144.

[11] D. R. McDonald (2005). The local limit theorem: A historical perspective. *Journal of The Iranian Statistical Society*, *4*(2), 73–86.

[12] S. D. Promislow (2014). *Fundamentals of Actuarial Mathematics*. John Wiley & Sons, West Sussex, UK.

[13] I. Shevtsova (2015). On convergence rate in the local limit theorem for densities. *Journal of Mathematical Sciences*, *205*(1), 105–112. https://doi.org/10.1007/s10958-015-2235-6.

[14] T. Siripraparat & K. Neammanee (2024). Discretized normal approximation for the number of descents. *Malaysian Journal of Mathematical Sciences*, *18*(4), 697–710. https://doi.org/10.47836/mjms.18.4.01.

[15] C. Stein (1972). A bound for the error in the normal approximation to the distribution of a sum of dependent random variables. In *Proceedings of the Sixth Berkeley Symposium On Mathematical Statistics and Probability, Volume 2: Probability Theory*, pp. 583–603. University of California Press, Durham, USA.

[16] J. K. Sunklodas (2014). On the normal approximation of a binomial random sum. *Lithuanian Mathematical Journal*, *54*(3), 356–365. https://doi.org/10.1007/s10986-014-9248-6.

[17] J. K. Sunklodas (2015). On the normal approximation of a negative binomial random sum. *Lithuanian Mathematical Journal*, *55*(1), 150–158. https://doi.org/10.1007/s10986-015-9271-2.